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# A TRIVIAL FORMALIZATION OF THE THEORY OF GROSSONE

**Abstract.** A trivial formalization is given for the informal reasonings presented in a series of papers by Ya. D. Sergeyev on a positional numeral system with an infinitely large base, grossone; the system which is groundlessly opposed by its originator to the classical nonstandard analysis.

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In recent years Ya. D. Sergeyev has published a series of papers [1–5] in which a positional numeral system is advanced related to the notion of grossone<sup>2</sup>. Ya. D. Sergeyev opposes his system to nonstandard analysis and regards the former as resting on different mathematical, philosophical, etc. doctrines. The aim of the present note is to properly position the papers by Ya. D. Sergeyev on developing numeral systems. It turns out that a model of Ya. D. Sergeyev’s system is provided by the initial segment  $\{1, 2, \dots, \nu!\}$  of the nonstandard natural scale up to the factorial  $\nu!$  of an arbitrary actual infinitely large natural  $\nu$ . Such a factorial serves as a model of Ya. D. Sergeyev’s grossone, thus demonstrating the place occupying by the numeral system he proposed.

As the main source we have chosen [4], the latest available paper by Ya. D. Sergeyev, which contains a detailed description of his basic ideas.

[4]: ... the approach used in this paper is different also with respect to the nonstandard analysis ... and built using Cantor’s ideas.

In the present note we are about to show that, contrary to what is expected by the author of [4], his indistinct definitions of grossone and the concomitant notions admit an extremely accurate and trivial formalization within the classical nonstandard analysis.

[4]: The infinite radix of the new system is introduced as the number of elements of the set  $\mathbb{N}$  of natural numbers expressed by the numeral  $\textcircled{0}$  called *grossone*.

Use the formalism of the internal set theory IST by E. Nelson [6] or any of the classical external set theories, for instance, EXT by K. Hrbáček [7] or NST by T. Kawai [8] (see also the monographs [9,10]). As usual,  ${}^\circ X$  denotes the standard core of a set  $X$ , i.e., the totality of all standard elements of  $X$ . In particular,  ${}^\circ \mathbb{N}$  is the totality of all finite (standard) naturals. Fix an arbitrary infinitely large natural  $\nu$  and denote its factorial by  $\textcircled{0}$ :

$$\textcircled{0} = \nu!, \quad \text{where } \nu \in \mathbb{N}, \nu \approx \infty.$$

Show that  $\textcircled{0}$  possesses all properties of “grossone” (postulated as well as implicitly presumed in [4]).

A possible approach to an adequate formalization (in the sense of [4]) of the notion of size or “the number of elements” of an arbitrary set  $A$  of standard naturals (i.e., of an external subset  $A \subset {}^\circ \mathbb{N}$ ) consists in assigning the natural  $\|A\| = |{}^*A \cap \{1, 2, \dots, \textcircled{0}\}|$  to each  $A$ , where  ${}^*A$  is the standardization of  $A$  and  $|X|$  is the size (in the usual sense) of a finite internal set  $X$ . In this case it is clear that  $\|{}^\circ \mathbb{N}\| = \textcircled{0}$ , which agrees with the fore-quoted “definition” of grossone. Note also that, due to the external induction, the function  $A \mapsto \|A\|$  possesses the additivity property (presumed in [4]):  $\|\bigcup_{k=1}^n A_k\| = \sum_{k=1}^n \|A_k\|$  for every family of pairwise disjoint sets  $A_1, \dots, A_n \subset {}^\circ \mathbb{N}$ ,  $n \in {}^\circ \mathbb{N}$ .

Another approach (which is more trivial and considerably closer to that of [4]) to defining the number of elements consists in “replacing” the set  ${}^\circ \mathbb{N}$  with the initial segment

$$\mathcal{N} = \{1, 2, \dots, \textcircled{0}\}$$

of the natural scale and considering the usual size  $|A| \in \mathbb{N}$  of each internal set  $A \subset \mathcal{N}$ . In this case, again,  $|\mathcal{N}| = \textcircled{0}$ ; and the additivity of the counting measure  $A \mapsto |A|$  needs no argument.

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<sup>2</sup> The term “grossone” belongs to Ya. D. Sergeyev, has no relevance to the usual meaning of the noun “gross” in English, and stems most likely from “groß” in German or “grosso” in Italian.

[4]: The new numeral  $\mathbb{O}$  allows us to write down the set,  $\mathbb{N}$ , of natural numbers in the form

$$\mathbb{N} = \{1, 2, 3, \dots, \mathbb{O} - 2, \mathbb{O} - 1, \mathbb{O}\}$$

because *grossone has been introduced as the number of elements of the set of natural numbers* (similarly, the number 3 is the number of elements of the set  $\{1, 2, 3\}$ ). Thus, grossone is the biggest natural number . . .

While crediting the author of [4] for the audacious extrapolation of the properties of the number 3, we nevertheless cannot accept the fore-quoted agreement if for no other reason than the fact that the set  $\mathbb{N}$  of naturals (in the popular sense of this fundamental notion) has no greatest element (with respect to the classical order). In order to keep the traditional sense for the symbol  $\mathbb{N}$  (and being governed by “Postulate 3. *The part is less than the whole*” of [4]), instead of reusing this symbol for the proper subset  $\{1, 2, \dots, \mathbb{O}\} \subset \mathbb{N}$  we decided to give the latter a less radical notation,  $\mathcal{N}$ .

[4]: *The Infinite Unit Axiom* consists of the following three statements:

*Infinity.* For any finite natural number  $n$  it follows  $n < \mathbb{O}$ .

*Identity.* The following relations link  $\mathbb{O}$  to identity elements 0 and 1

$$0 \cdot \mathbb{O} = \mathbb{O} \cdot 0 = 0, \quad \mathbb{O} - \mathbb{O} = 0, \quad \frac{\mathbb{O}}{\mathbb{O}} = 1, \quad \mathbb{O}^0 = 1, \quad 1^{\mathbb{O}} = 1, \quad 0^{\mathbb{O}} = 0.$$

*Divisibility.* For any finite natural number  $n$  sets  $\mathbb{N}_{k,n}$ ,  $1 \leq k \leq n$ , being the  $n$ th parts of the set,  $\mathbb{N}$ , of natural numbers have the same number of elements indicated by the numeral  $\frac{\mathbb{O}}{n}$ , where

$$\mathbb{N}_{k,n} = \{k, k + n, k + 2n, k + 3n, \dots\}, \quad 1 \leq k \leq n, \quad \bigcup_{k=1}^n \mathbb{N}_{k,n} = \mathbb{N}.$$

Since  $\mathbb{O} = \nu!$  is an infinitely large number, it satisfies *Infinity*. Every natural meets *Identity*, and so does  $\mathbb{O}$ . Presenting the factorial of an infinitely large number,  $\mathbb{O}$  is divisible by every standard natural. Moreover, if  $n \in {}^\circ\mathbb{N}$ ,  $1 \leq k \leq n$ , and

$$\begin{aligned} {}^\circ\mathbb{N}_{k,n} &= \{k + (m - 1)n : m \in {}^\circ\mathbb{N}\}, \\ \mathcal{N}_{k,n} &= \mathcal{N} \cap \{k + (m - 1)n : m \in \mathbb{N}\}; \end{aligned}$$

then  $\|{}^\circ\mathbb{N}_{k,n}\| = |\mathcal{N}_{k,n}| = \frac{\mathbb{O}}{n}$ . Hence,  $\mathbb{O}$  meets *Divisibility*.

[4]: It is worthy to emphasize that, since the numbers  $\frac{\mathbb{O}}{n}$  have been introduced as numbers of elements of sets  $\mathbb{N}_{k,n}$ , they are integer.

If a number is declared natural, it naturally cannot occur unnatural. To remove all doubts, we suggest a rigorous and detailed justification for satisfiability of the above postulate: for every  $n \in {}^\circ\mathbb{N}$  we have  $n < \nu$  and thus

$$\text{the number } \frac{\mathbb{O}}{n} = \frac{\nu!}{n} = \frac{1 \cdot 2 \cdot \dots \cdot n \cdot \dots \cdot \nu}{n} \text{ is integer.}$$

- [4]: The introduction of grossone allows us to obtain the following interesting result: the set  $\mathbb{N}$  is not a monoid under addition. In fact, the operation  $\textcircled{0} + 1$  gives us as the result a number grater than  $\textcircled{0}$ . Thus, by definition of grossone,  $\textcircled{0} + 1$  does not belong to  $\mathbb{N}$  and, therefore,  $\mathbb{N}$  is not closed under addition and is not a monoid.

Indeed,  $\textcircled{0} \in \{1, 2, \dots, \textcircled{0}\} = \mathcal{N}$ , but  $\textcircled{0}+1 \notin \{1, 2, \dots, \textcircled{0}\} = \mathcal{N}$ . (However, taking it into account that  $\mathcal{N}$  is not the set of all naturals, the above trivial observation is unlikely “interesting.”)

- [4]: ... adding the Infinite Unit Axiom to the axioms of natural numbers defines the set of extended natural numbers indicated as  $\widehat{\mathbb{N}}$  and including  $\mathbb{N}$  as a proper subset

$$\widehat{\mathbb{N}} = \{1, 2, \dots, \textcircled{0} - 1, \textcircled{0}, \textcircled{0} + 1, \dots, \textcircled{0}^2 - 1, \textcircled{0}^2, \textcircled{0}^2 + 1, \dots\}.$$

In fact,  ${}^\circ\mathbb{N}$  and  $\mathcal{N}$  are both proper subsets of the set  $\mathbb{N}$  of *all* naturals. (As is known, the radical formalism of IST saves us from considering “extended numbers.”)

We permit ourselves to pass over other numerous descriptions of the properties of grossone and the accompanying notions in [4], since the corresponding analysis is quite analogous to that above (and equally trivial). However, we cannot help commenting the declared elimination of Hilbert’s paradox of the Grand Hotel:

- [4]: ... it is well known that Cantor’s approach leads to some “paradoxes” ... Hilbert’s Grand Hotel has an infinite number of rooms ... If a new guest arrives at the Hotel where every room is occupied, it is, nevertheless, possible to find a room for him. To do so, it is necessary to move the guest occupying room 1 to room 2, the guest occupying room 2 to room 3, etc. In such a way room 1 will be available for the newcomer ...

... In the paradox, the number of the rooms in the Hotel is countable. In our terminology this means that it has  $\textcircled{0}$  rooms ... Under the Infinite Unit Axiom this procedure is not possible because the guest from room  $\textcircled{0}$  should be moved to room  $\textcircled{0} + 1$  and the Hotel has only  $\textcircled{0}$  rooms. Thus, when the Hotel is full, no more new guests can be accommodated — the result corresponding perfectly to Postulate 3 and the situation taking place in normal hotels with a finite number of rooms.

The following unpretentious “paradox of the Gross Hotel” is brought to the audience’s attention: Even though all the grossrooms  $1, 2, \dots, \textcircled{0}$  are occupied, it is easy to accommodate one more client in the Gross Hotel. To this end it suffices to move the guest occupying room  $n$  to room  $n + 1$  for each *finite*  $n$ . Since  $n + 1 < \textcircled{0}$  for all finite  $n$ , all the former guests get their rooms in the Gross Hotel, while room 1 becomes free for a newcomer.

Besides a babbling theorization around grossone, [4] includes an “applied” part dedicated to a new positional numeral system with base  $\textcircled{0}$ . (The system is meant for becoming a foundation for “Infinity Computer” [5] which is able to operate infinitely large and infinitesimal numbers.) Unfortunately, the corresponding exposition remains highly informal, and even crucial definitions are substituted with allusions and illustrating examples.

- [4]: In order to construct a number  $C$  in the new numeral positional system with base  $\mathbb{O}$  we subdivide  $C$  into groups corresponding to powers of  $\mathbb{O}$ :

$$C = c_{p_m} \mathbb{O}^{p_m} + \cdots + c_{p_1} \mathbb{O}^{p_1} + c_{p_0} \mathbb{O}^{p_0} + c_{p_{-1}} \mathbb{O}^{p_{-1}} + \cdots + c_{p_{-k}} \mathbb{O}^{p_{-k}}.$$

. . . Finite numbers  $c_i$  are called *infinite grossdigits* and can be both positive and negative; numbers  $p_i$  are called *grosspowers* and can be finite, infinite, and infinitesimal (the introduction of infinitesimal numbers will be given soon). The numbers  $p_i$  are such that  $p_i > 0$ ,  $p_0 = 0$ ,  $p_{-i} < 0$  and

$$p_m > p_{m-1} > \cdots > p_2 > p_1 > p_{-1} > p_{-2} > \cdots > p_{-(k-1)} > p_{-k}.$$

. . . Finite numbers in this new numeral system are represented by numerals having only one grosspower equal to zero . . .

. . . all grossdigits  $c_i$ ,  $-k \leq i \leq m$ , can be integer or fractional . . . Infinite numbers in this numeral system are expressed by numerals having at least one grosspower greater than zero . . . Numerals having only negative grosspowers represent infinitesimal numbers.

In the fore-quoted definitions, combinations of the terms “finite,” “infinite,” and “number” seem to be used quite vaguely. For instance, it is unclear from the text whether a numeral is assumed infinite (and in what sense) if it is not finite (in some sense). Following the definitions of [4] literally, a grosspower can be finite, infinite, and (or?) infinitesimal, while “finite” means  $c\mathbb{O}^0$  (a grossdigit  $c$ , a rational numeral), “infinite” is expressed by a numeral having at least one strictly positive grosspower, and “infinitesimal” is a numeral whose grosspowers are all strictly negative. Seemingly, this implies that a grosspower cannot be equal to, say,  $\mathbb{O}^0 + \mathbb{O}^{-1}$ , but the subsequent examples of [4] show that this is not so, and arbitrary numerals can serve as grosspowers. In addition, the reason is completely unclear for choosing the terms “infinite” and “infinitesimal” exactly for the classes of numerals mentioned in the quote. For instance, the numeral  $a = \mathbb{O}^{\mathbb{O}^{-1}}$  (with grosspower  $\mathbb{O}^{-1} > 0$ ) is “infinite” by definition, while, obviously,  $1 < a < 2$ . On the other hand, the numeral  $b = \mathbb{O}^{\mathbb{O}^{-1}} - 1$  is also considered “infinite” and not “infinitesimal,” while, as is easily seen,  $b$  is infinitely close to zero in the sense that  $-c < b < c$  for every finite  $c > 0$ .

Regardless of terminological discipline, the fore-quoted definition of numerals  $C$  cannot be considered formal if for no other reason than the participating notion of (“infinite” and “infinitesimal”) grosspowers depends on the initial notion of numeral, thus leading to a vicious circle. In addition, from the illustrations of [4] it is clear that the positional system proposed admit syntactically different numerals with coincident values: for instance,  $0\mathbb{O}^0 \equiv 0\mathbb{O}^1$ ,  $1\mathbb{O}^0 \equiv 1\mathbb{O}^{0\mathbb{O}^0}$ . (The notion of the value of a term and the equivalence relation  $\equiv$  are clarified in [11].) At the same time, [4] misses not only the corresponding stipulations (easy to guess though) but also any attempts of justifying the unambiguity of the positional system, even under implicit stipulations. Observe also that the description of [4] for the algorithms of calculating the sum and product of numerals (i.e., of finding the corresponding equivalent numeral) is very superficial, since it does not touch upon the problem of recognizing equivalent numerals (which is necessary for collecting similar terms) and that of comparing them (which is necessary for collating the summands in order of their “grosspowers”). It is thus not surprising that the patent application [5] reports on the development of “Infinity Calculator” which is able to handle numerals admitting “finite exponents” only.

To provide some justification, we briefly described in [11] one of the possible approaches to formalization of the notion of numeral as well as the corresponding algorithmic procedures.

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